How big was that huge volcanic eruption which happened 90 thousand years ago?

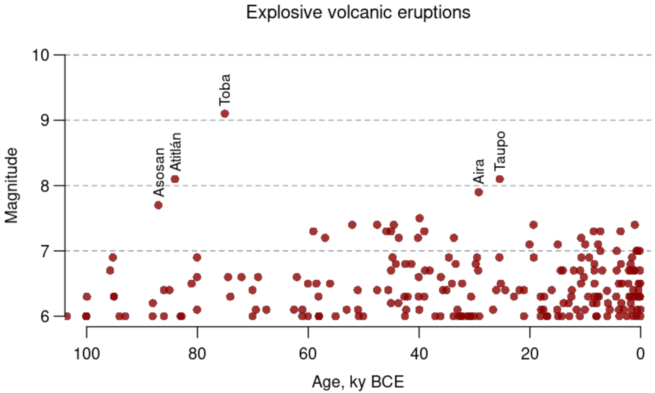
Jonathan Rougier

Rougier Consulting Ltd
& University of Bristol

Jointly with Steve Sparks, Willy Aspinall, Sue Mahony

Representing and quantifying uncertainty in complex systems
RSS Annual Conference, Sep 2021

Large explosive volcanic eruptions



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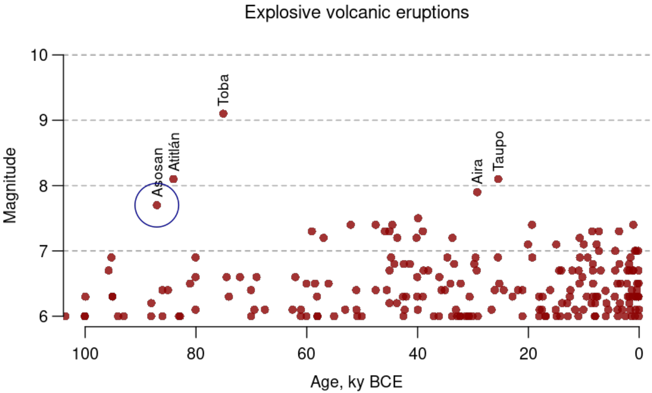

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Aso volcano today

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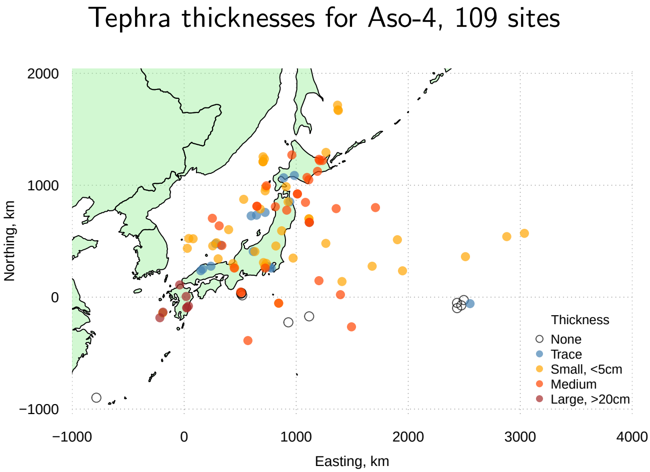

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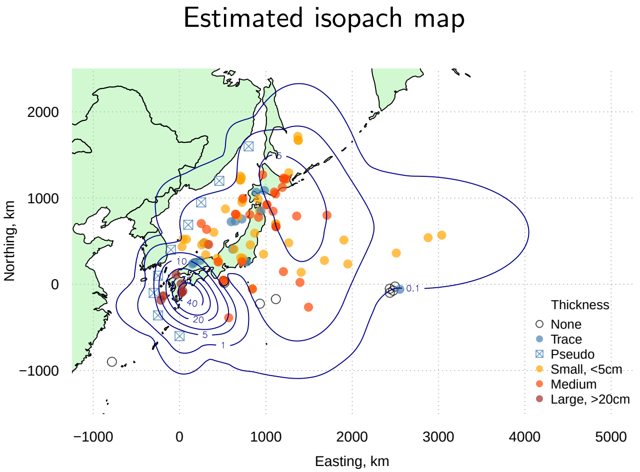

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- Current practice for volume estimation involves a high degree of expert judgement, being based on a small number of hand-drawn isopachs (= contours of equal thickness).

  The interplay of measurement and judgement is not transparent, and the approach is not amenable to a formal assessment of uncertainty.
- We use modern flexible-fitting methods from Statistics and Machine Learning, we represent judgements using pseudo-measurements, and we quantify variability using a bootstrap 95% CI.

A spatial model for thickness

We insist from the outset that thickness z is non-negative:

$$\mathrm{BC}(z(s);\lambda) = \mathrm{BC}(0;\lambda) + \sum_{j=1}^{k} \beta_j \phi_j(s) + r(s), \quad s \in \mathcal{S} \subset \mathbb{R}^2,$$

where $\phi_j \geq 0$ with compact support, $\beta_j \geq 0$, and 'BC' is the Box-Cox transformation,

$$\mathrm{BC}(z;\lambda) = \begin{cases} \log(z+1) & \lambda = 0 \\ \frac{(z+1)^\lambda - 1}{\lambda} & \lambda \neq 0. \end{cases}$$

- This model is a bit bespoke, but we need $\hat{z}(s) \geq 0$ everywhere, *and* $\hat{z}(s) = 0$ for all locations outside the footprint of $\bigcup_j \phi_j$, regardless of how we transform z.

Modelling options

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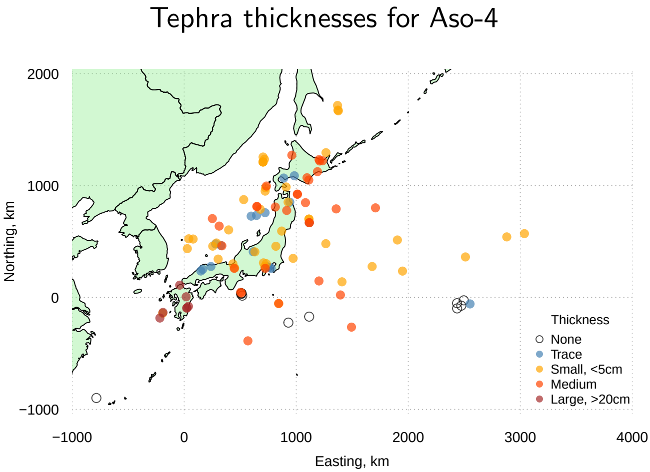
Let's take these in reverse order ...

Modelling options, pseudo-measurements



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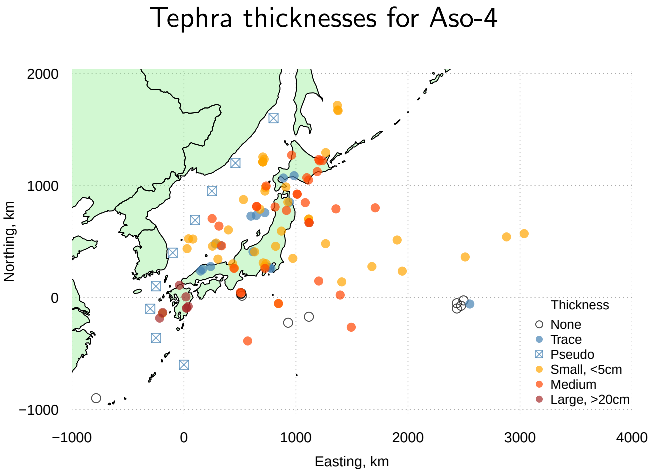

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- So after some experimentation, we have currently settled on $\lambda = \frac{1}{2}$ (square-root), which seems to give reasonable results, as judged by the volcanologists.

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- But then map the unit disk to ellipses,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

  and use a lot of them.

Modelling options, basis functions

- Where to put these ellipses?
  - Putting them at the data sites leads to over-fitting,
  - And yet we want more where the density of sites is high,
  - So I ended up using the Delaunay triangulation of the sites.
  - Specifically, the centroids of the Delaunay tiles with minimum edge length of at least 100 km.

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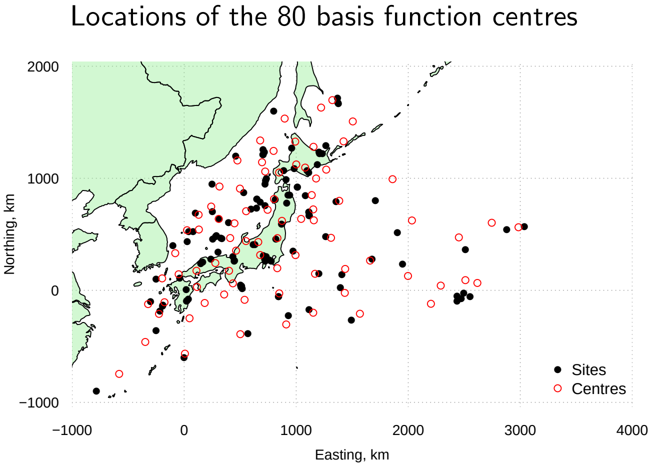
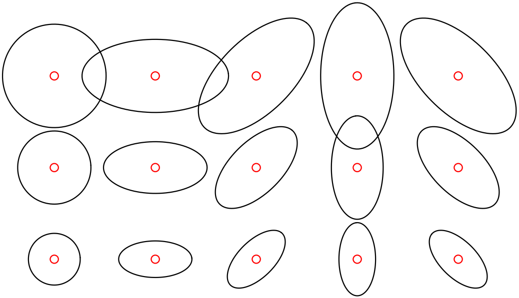


Modelling options, basis functions

Multi-resolution: I put 15 ellipses at every centre,



where the area of the largest subset (top) was treated as a bandwidth parameter, to be learnt and plugged-in; $15 \times 80 = 1200$ basis functions altogether. 😳

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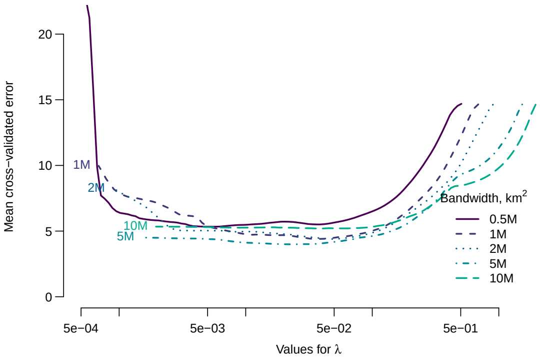
- Downweight the 'trace' measurements using $w_i = 1/4$
- Fit with an L_1 penalty, and use ten-fold cross-validation to choose both the sparsity coefficient λ and the bandwidth:



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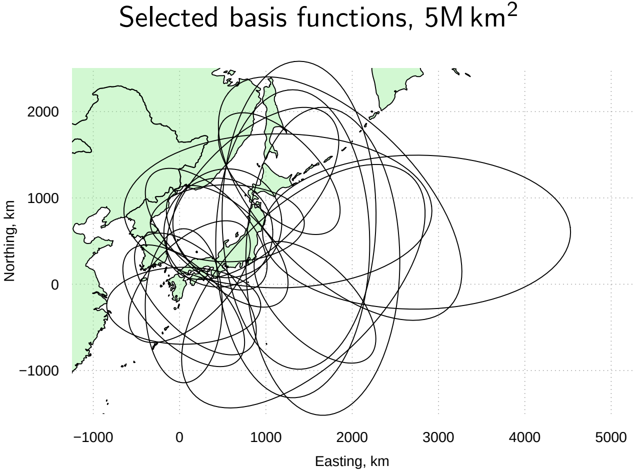

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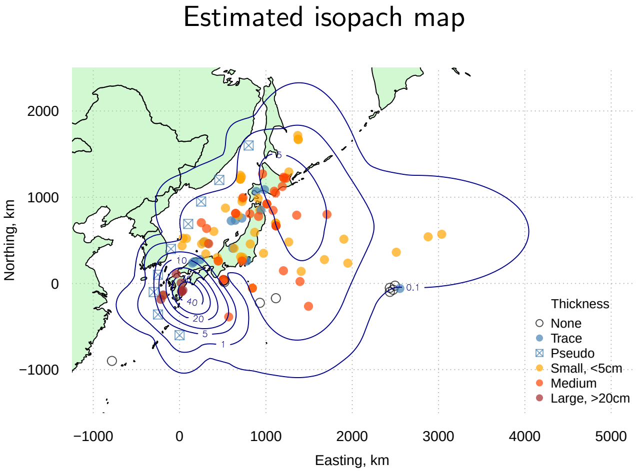

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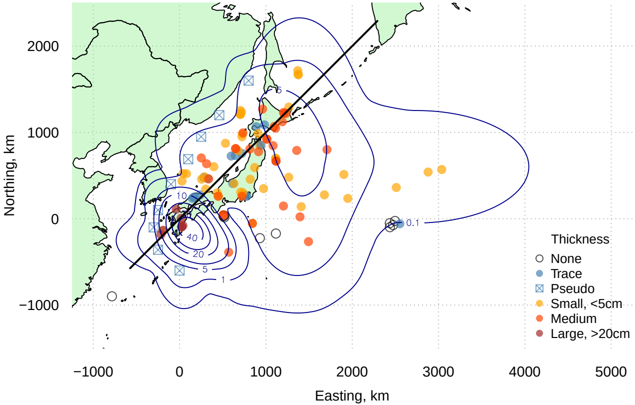
Along the transect



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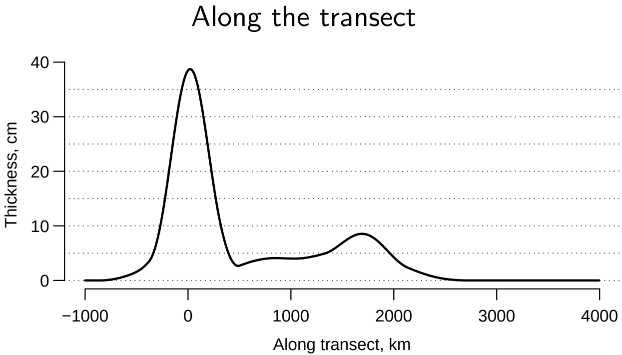
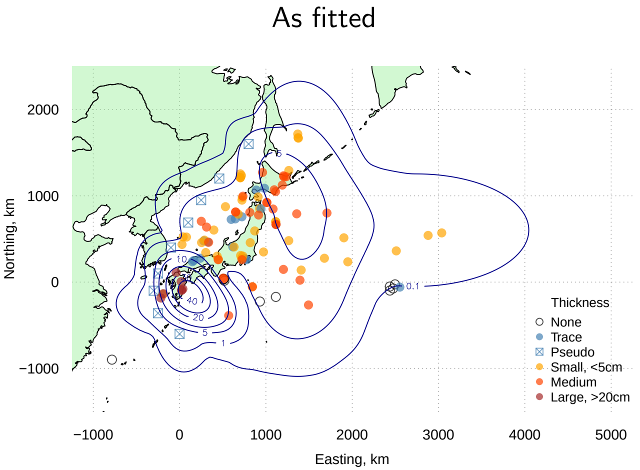


The effect of the pseudo-measurements



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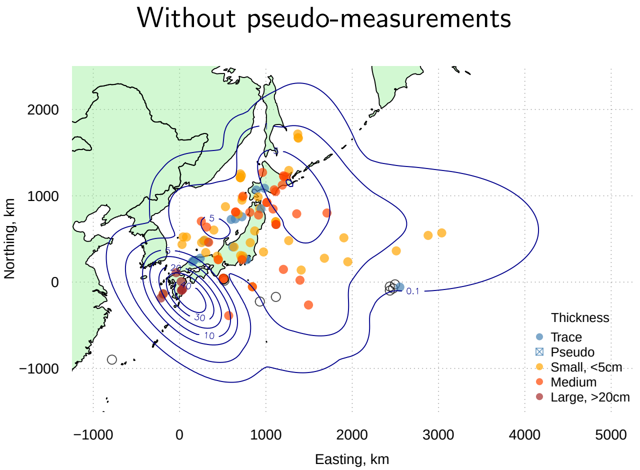

Estimating volume

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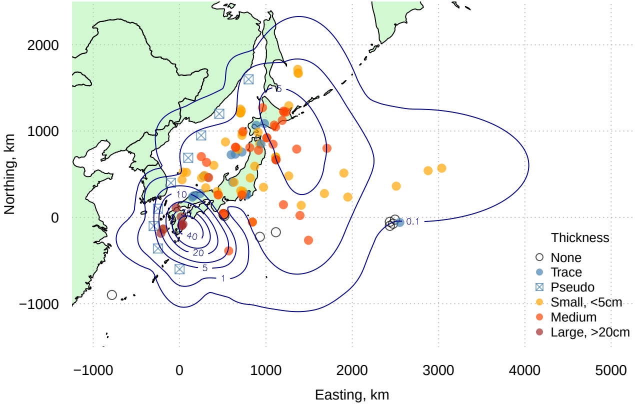

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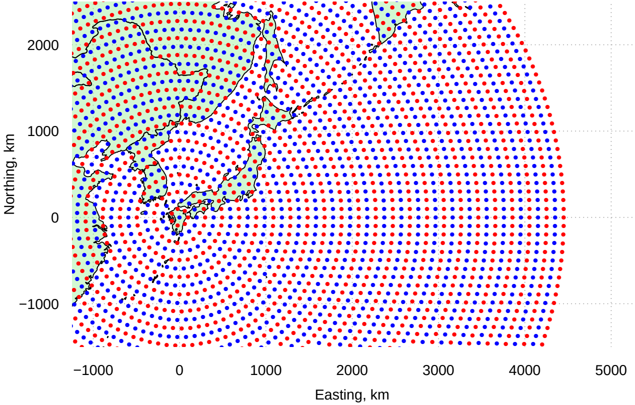

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4. The resulting 95% CI is $[220\,\mathrm{km}^3, 370\,\mathrm{km}^3]$.

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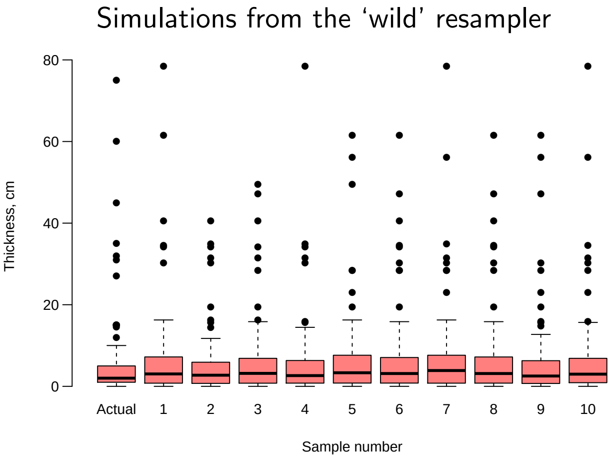
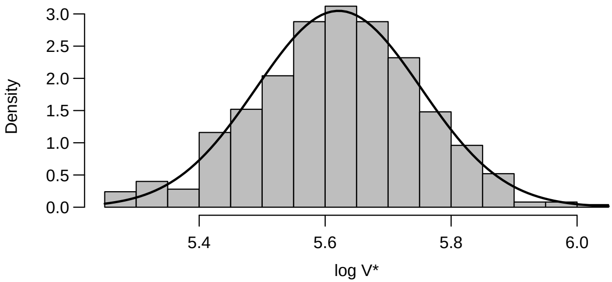


The slightly lower IQR for the 'Actual' is due to treatment of the 'trace' measurements; has negligible effect on the volume estimate.

What about variability? (cont)

Histogram of replicates of log-volume



This looks OK – thank goodness. Originally I used the variance-stabilized Studentized Pivotal Bootstrap, but it gave roughly the same 95% CI.

Take-home messages

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3. Modern methods from Statistics and Machine Learning can provide flexible and data-driven methods for constructing an algorithm for transforming a dataset into an estimate.

And don't be surprised if the resulting 95% CI is quite large!

References

Davidson, R. and Flachaire, E. (2008). The wild bootstrap, tamed at last. *Journal of Econometrics*, 146:162–169.